

Two body scattering length of Yukawa model on a lattice

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The extraction of scattering parameters from Euclidean simulations of a Yukawa model in a finite volume with periodic boundary conditions is analyzed both in non relativistic quantum mechanics and in quantum field theory.

1. Introduction

A rigorous treatment of nuclei based on first principles quantum chromodynamics requires the understanding of the transition from the confining theory of quarks to the unconfined hadrons, a present challenge for the high energy community. Only very recently, the development of algorithms and computers devoted to lattice QCD has made possible starting the study of nucleon-nucleon [1] systems (as well as $\pi\pi$, KK , πN , etc) based on full QCD calculations. Nevertheless, lattice methods have a drawback for computing scattering parameters, as they can not be extracted in the infinite volume limit [2]. However they can be obtained by measuring the finite volume effects on the spectrum of a periodic boundary condition box [3].

In this work, the scattering length of the Yukawa model has been determined; on one hand, for the Yukawa potential $V(r) = -\alpha e^{-m_s r}/r$ in Schrodinger equation, that can be precisely solved in a 3-d torus, as well as in the infinite volume, serving as a crosscheck for determining scattering length via finite volume effects. On the other hand, the quantum field theory of fermions (ψ) and mesons (ϕ) interacting via the lagrangian $\mathcal{L}_I(x) = g_0 \bar{\psi}(x) \Gamma \phi(x) \psi(x)$ – that gives rise to the latter potential (with $\alpha = g_0^2/4\pi$) in the NR limit – is solved in a Euclidean lattice.

The goals of this calculation are manyfold. First, the comparison between the QFT and the non relativistic (NR) results can help to disentangle the bias introduced by the ladder approximation, as well as the size of the relativistic corrections. Second, the methods used to extract the scattering length can be tested with this model, much simpler than QCD. And finally, the interaction itself is interesting, as scalar meson exchange is a common ingredient of NN interaction potentials [4], responsible for most of the nuclear binding energy.

2. From bound state spectrum to low energy parameters

The relation between the spectrum of a system enclosed in a torus and the phase-shifts in the infinite volume has been established by M. Luscher [3]. For a lattice with L points in each spacial direction and spacing a , the energies are determined by the expression:

$$p \cot(\delta(p)) = \frac{1}{\pi a L} S \left(\left(\frac{a L p}{2\pi} \right)^2 \right), \quad S(\eta) = \lim_{\Lambda \rightarrow \infty} \sum_{\vec{n} \in Z^3}^{\|\vec{n}\| \leq \Lambda} \frac{1}{n^2 - \eta} - 4\pi\Lambda. \quad (1)$$

that is exact for lattice sizes $aL > 2R$ where R is the interaction range, i.e. $V(x > R) = 0$. For interactions of physical interest this regime is reached exponentially and independently of the coupling constant and the low energy parameters (LEP) appearing in the effective range expansion, $p \cot(\delta(p)) \approx -\frac{1}{a_0} + \frac{1}{2}r_0 p^2 + \dots$. This constitutes a remarkable advantage with respect to the, most commonly used, large- L expansion of (1):

$$E \approx \frac{4\pi a_0}{M(La)^3} \left(1 + c_1 \frac{a_0}{aL} + c_2 \left(\frac{a_0}{aL} \right)^2 + \dots \right). \quad (2)$$

In a recent work [5], the possibility to extract the LEP from the bound state spectra in a finite box has been examined in the framework of the NR Yukawa model. The L -dependence of the eigenenergies $E(L)$ in a 3d torus was used to extract the infinite volume scattering length a_0 and effective range r_0 . Contrary to the quantum field lattice calculation, those quantities can be independently computed by solving the corresponding Schrodinger radial equation and used to test the validity of the different approaches discussed in the literature.

The results were analyzed in terms of the slowly varying quantity $a_0^{(0)}(L) \equiv (aL)^3 E(L) M / 4\pi$ which represents the zeroth-order approximation of large- L expansion (2). The NR model depends on the unique parameter $G = \alpha M / m_s$, and it was shown that $a_0^{(0)}(L) m_s$ tends to $-G$ in the small- L limit, and – according to (2) – to the required scattering length value a_0 for $aL \rightarrow \infty$ (figure 1). It is interesting to remark that $a_0^{(0)}(L)$ displays the same $1/L^3$ behavior in both limits.

It was in particular found that one can obtain accurate values of a_0 and r_0 by using equation (1) at two different lattice sizes in the region $Lam_s \approx 5$ and solving the resulting linear system. Some results are illustrated in figure 2. They show the sensibility to the effective range r_0 and the need for both parameters to be simultaneously determined.

The possibility of extracting LEP that way is independent of the a_0 value and applies in the resonant case as well. Care must be taken however when applying equation (2) to extract the value of a_0 . When using lattice sizes $Lam_s \sim 2-3$, $E(L)$ is well fitted by a c/L^3

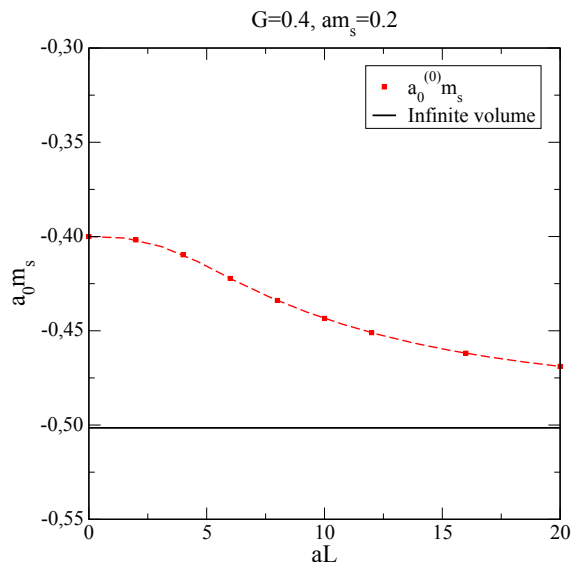


Figure 1: Lattice result for $a_0^{(0)}$ vs volume compared to the infinite volume result.

dependence but the coefficient c can strongly differ from the infinite volume scattering length [5]. Only the use of equation (1) at lattice sizes greater than the interaction range – $Lam_s \gtrsim 5$ in the Yukawa model – could lead to unambiguous extraction of LEP, provided both a_0 and r_0 are taken into account.

3. Lattice results

The non perturbative solutions of the Yukawa field theory in the lattice have been considered in [6, 7]. The existence of a critical value of the coupling has been argued [7, 8] hindering the appearance of bound states of the QFT. The effect of the ladder approximation can nevertheless be studied by calculating the scattering length and comparing with the non relativistic results.

We have used Wilson discretization of fermion fields, which has the disadvantage that there are $\mathcal{O}(a)$ discretization errors. They can be important in our case, since nucleon mass is not small. As the interaction is non confining, to reduce the discretization errors we will use a hint from the free case, where the time evolution of fermion is given by

$$C(t) = \sum_{\vec{x}} Tr \langle \bar{\psi}(\vec{x}, t) \psi(\vec{0}, 0) \rangle \propto (1 + aM_L)^{-t} \sim e^{-Mt}. \quad (3)$$

This expression defines the mass that governs the time evolution of the nucleon M , and relates it to the fermion mass appearing in the lagrangian, M_L , as $aM_L = \exp(aM) - 1$.

Both masses are equivalent at the continuum limit, but they can differ sizably when working at finite lattice spacing, what may spoil the convergence of the lattice results. To overcome this, we define an improved coupling, \tilde{g} , as the variation of nucleon energy with the meson field at zero meson field (note that the coupling acts as a mass term),

$$\tilde{g} = \left. \frac{\partial E_N(\vec{p}=0)}{\partial \phi^b} \right|_{\phi^b=0}. \quad (4)$$

If we assume that the nucleon propagates in a small constant background field, ϕ_b , its mass is modified by the coupling term as $aM_L \rightarrow aM_L + g_0\phi_b$ which implies $aM \rightarrow aM + g_0\phi_b/(1 + aM_L)$. The renormalized coupling defined above results then

$$\tilde{g} = \frac{g_0}{1 + aM}. \quad (5)$$

Note that g_0 and \tilde{g} are equivalent in the continuum limit ($a \rightarrow 0$), but the second should have a better behavior at finite lattice spacing. The usefulness of this coupling is specially

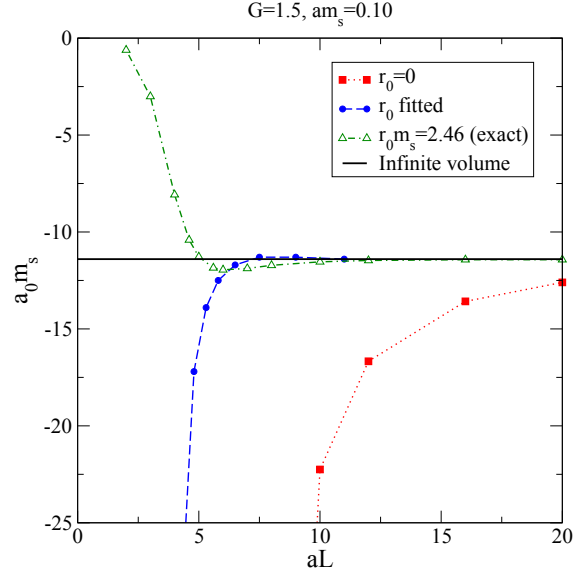
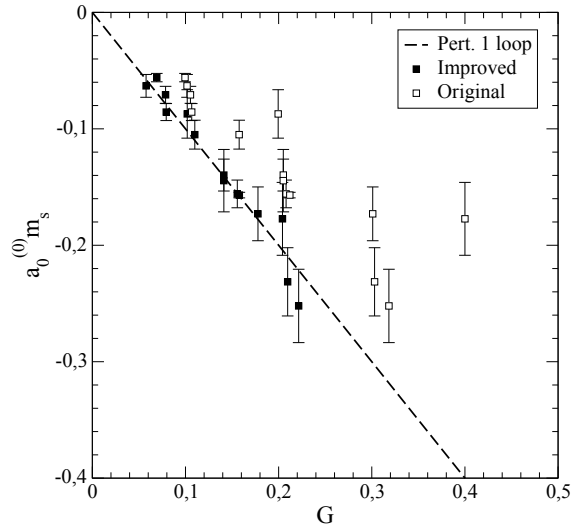


Figure 2: a_0 obtained with equation (1) for different values of r_0 . The influence of this parameter is remarkable.

important when working with coarse lattices, as is the case in nuclear physics, due to the mass hierarchy. Note that the differences due to this definition of the coupling can be very important, $\sim 70\%$ for fermion masses $aM \sim 0.3$.

Two body S-wave scattering length can be computed from the corresponding spectrum on a finite size lattice calculation as it has been shown in the NR potential model. The first test to be done is the perturbative regime that, as in the NR case, predicts for the scattering length $a_0 m_s = -G + \mathcal{O}(G^2)$. The results of the scattering length obtained from lattice calculations are shown in figure 3 as a function of the coupling constant $\alpha = g^2/4\pi$, both for the original and the improved coupling. The correspondence with the small coupling limit (dashed line) gets clearly better with the use of the improvement factor (4).



The fact that we do not see differences with the first order in perturbation theory is maybe due to the fact that we work with rather reduced lattice volumes. Further calculations have to be done (in particular bigger volumes with $La \gtrsim 5m_s^{-1}$) in order to compute the scattering length – and effective range – properly.

Figure 3: Scattering length versus G for the original g_0 and modified coupling \tilde{g} for lattice volumes $La m_s \sim 2 - 4$.

As a concluding remark, we have proposed an improved definition of Yukawa coupling constant in the lattice that exhibits quite better scaling properties than the bare one. As shown in figure 3, this correction factor is mandatory to test perturbation theory – and accordingly to obtain any physical result beyond this regime – even for rather small lattice spacings $aM \sim 0.2$.

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